Ergodic theory of generic continuous maps

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Abstract

We study the ergodic properties of generic continuous dynamical systems on compact manifolds. As a main result we prove that generic homeomorphisms have convergent Birkhoff averages under continuous observables at Lebesgue almost every point. In spite of this, when the underlying manifold has dimension greater than one, generic homeomorphisms have no physical measure — a somewhat strange result which stands in sharp contrast to current trends in generic differentiable dynamics. Similar results hold for generic continuous maps.

To further explore the mysterious behaviour of C^0 generic dynamics, we also study the ergodic properties of continuous maps which are conjugated to expanding circle maps. In this context, generic maps have *divergent* Birkhoff averages along orbits starting from Lebesgue almost every point.

Keywords: ergodic theory, physical measures, genericity, circle homeomorphisms.

MSC 2000: 37A99.

1 Introduction

One of the best-known results in ergodic theory, due to Ulam-Oxtoby [OU], which in fact gave birth to Baire Category arguments in dynamics, states that generic volume preserving homeomorphisms on a compact manifold are ergodic. Although seven decades have passed, a dissipative (i.e., non-volume preserving) analogue of their theorem has still not appeared. This is not because of a lack of interest in the dynamics of generic homeomorphisms, which is in fact a very active area of research in dynamical systems, treated extensively in the works of Alpern and Prasad [AP] and Akin et. al. [AHK]. While Alpern and Prasad consider the ergodic theory of generic volume

preserving homeomorphisms, Akin et. al. study topological properties in the generic dissipative case. We blend the two approaches and consider ergodic properties of generic dissipative homeomorphisms. In order to do that, we must first decide what we consider to be the appropriate questions to ask. In the differentiable setting it has long been more or less clear what these should be: one should ask whether a generic diffeomorphism has some (possibly many) *physical measures* capturing the statistical behaviour of most orbits. We have found that applying the same type of questions to generic homeomorphisms leads to fashinating insights.

Physical measures have been much in vogue ever since they were introduced by Sinai, Ruelle, and Bowen in the 70's and shown to exist for every C^2 Axiom A diffeomorphism [Rue1]. No robust obstacle to the existence of physical measures is known in differentiable dynamics, which is quite generally believed to be a C^r dense phenomenon [Pal]. Some doubt, however, has been expressed by Ruelle [Rue2]. He seriously considers the possibility of some robust mechanism that provides *historical behaviour* — his term for the lack of convergence of Birkhoff averages on a set of positive Lebesgue measure (a phenomenon beautifully illustrated in a famous example due to Bowen).

The current work fills the vacuum left after the result of Ulam-Oxtoby by proving that, in the context of generic homeomorphisms, no historical behaviour exists.

Theorem. *For a generic homeomorphism f of any compact manifold M, the Birkhoff averages*

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k(x))$$

of every continuous $\varphi : M \to \mathbb{R}$ are convergent Lebesgue almost everywhere.

We also reveal a surprising cenario, very different from that expected in the differentiable setting: although Birkhoff averages exist Lebesgue almost everywhere, a generic homeomorphism has no physical measure (except for the very special case where the underlying manifold is the circle). The lack of physical measures in this context is not at all due to historical behaviour, but to an entirely different phenomenon, unconceivable in the context of generic differentiable dynamics. It is the same phenomenon that appears in the differentiable world in very rigid situations such as the identity map and rational translations of the torus: Birkoff averages exist, but any two points do (Lebesgue) almost surely have different ones.

2 Some background

As mentioned in the introduction, the theory of C^0-generic dynamics has been extensively studied from the topological viewpoint, mainly by M. Hurley and extended in the book [AHK]. It is known that generic continuous maps (and in particular generic homeomorphisms) of sufficiently regular topological spaces – say compact manifolds – have highly complicated and even pathological dynamics from the topological point of view. For instance, for generic homeomorphisms (i) the nonwandering set is a Cantor set [AHK] which contains a dense subset of periodic points [Hur1], (ii) any Baire-residual point in the manifold has an adding machine as both its omega- and alpha-limit sets [AHK], and (iii) there are infinitely many topologically repelling sets and infinitely many topologically attracting sets, infinitely nested within each other: every attractor contains repellors, and vice-versa [Hur2].

In the introduction to their delightful book, Hurley et al ask whether their results admit ergodic analogues: "the question of whether something analogous to our results can be obtained in the measure theoretic category is an open one" (page 1). They later discuss how to approach this question, and suggest the use of Lebesgue measure: "while there are difficulties in finding an appropriate measure on the space of homeomorphisms, at least on a manifold Lebesgue measure is certainly appropriate in the context of questions on the behavior of "most points"" (page 5).

In this paper we follow their suggestion: instead of looking at topologically dynamical properties of a Baire-residual set of points, we examine ergodic-theoretic properties of a full-Lebesgue-measure set of points. We show in various contexts that the dynamics of generic continuous maps are indeed also very pathological (*weird* or *wacky* or, sometimes, even *wicked* – see Definition 3.4 below) when viewed from this ergodic perspective.

3 The Results

Before we state our results we shall provide some vocabulary and notation. We begin with notation for the spaces we work in and an explanation of what is meant by "generic".

Throughout this paper M denotes a compact connected boundaryless manifold. We denote by $C^0(M)$ the space of all continuous maps from M to itself and by $\mathrm{Homeo}(M)$ the space of all homeomorphisms of M to itself.

Both of these spaces are endowed with the usual C^0 metric

$$d_{C^0}(f,g) = \sup_{x\in M} d(f(x), g(x)). \tag{1}$$

In so doing, the space $C^0(M)$ becomes a complete metric space; however, the space $\mathrm{Homeo}(M)$ does not. However, there is another metric on $\mathrm{Homeo}(M)$ defined by

$$d_{\mathrm{Hom}}(f,g) = d_{C^0}(f,g) + d_{C^0}(f^{-1}, g^{-1}). \tag{2}$$

This metric is complete and generates the same topology as d_{C^0} does — the C^0 topology. In practice we shall only use the metric d_{C^0}, simply denoted by d.

A subset $\mathcal{R}$ of a topological space X is *residual* if it contain the intersection $\bigcap_{k\in\mathbb{N}} V_k$ of a countable family of open-and-dense subsets V_k of X. A topological space X is a *Baire space* if every residual subset of X is dense in X. By the Baire Category Theorem, every complete metric space is Baire. In particular the spaces $C^0(M)$ and $\mathrm{Homeo}(M)$ are Baire with respect to the C^0 topology.

Definition 3.1. *A property $\mathcal{P}$ is said to be* generic *in the space X if there exists a residual subset $\mathcal{R}$ of X such that every element $p \in \mathcal{R}$ satisfies property $\mathcal{P}$.*

Note that, given a countable family of generic properties $\mathcal{P}_1, \mathcal{P}_2, \ldots$, all of the properties $\mathcal{P}_i$ are *simultaneously* generic in X. This is because the family of residual sets is closed under countable intersections.

Now for some ergodic definitions and notation. By "measure" we always refer to non-signed measures defined on the Borel σ-algebra of the ambient manifold M. We denote by $\mathcal{M}(M)$ the set of all probability measures on M, and by $\mathcal{M}_f(M)$ the set of all f-invariant probabilities on M. Both of these spaces are endowed with the usual weak* topology, turning them into compact metrizable spaces. We fix and denote by m a volume probability on M which we simply refer to as "Lebesgue measure" (see remark **??** regarding the relevance of volume measures).

Given a continuous dynamical system $f : M \to M$ and a point $x \in M$, the *Birkhoff limit* of the point x, when it exists, is given by the probability measure $\mu_x = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \delta_{f^j(x)}$, where δ_y denotes the one-point Dirac probability supported on y and the limit is taken in the weak* topology. When this limit exists, it is a fortiori an f-invariant probability. The Birkhoff

limit μ_x is characterized by the following condition: given any continuous function $\varphi : M \to \mathbb{R}$, the average $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \varphi(f^j(x))$ coincides with the integral $\int_M \varphi \, d\mu_x$.

A map f is *totally singular* (with respect to Lebesgue measure) if there is a (Borel) measurable set Λ such that $m(\Lambda) = 1$ and $m(f^{-1}(\Lambda)) = 0$.

Definition 3.2. *Given a Borel probability μ on M we define its basin to be the set*

$$B(\mu) = \{x \in M : \mu_x \text{ is well defined and coincides with } \mu\}.$$

A probability μ is called a physical measure if $B(\mu)$ has positive Lebesgue measure.

Remark 3.3. Only invariant measures can have non-empty basin. In particular, every physical measure is invariant. It is not true, however, that physical measures are necessarily ergodic (see [MYNPV] for a counterexample).

Given a periodic point p of period π, its orbit supports a unique invariant probability, the *periodic Dirac measure*, given by $\mu_p = \frac{1}{\pi}\sum_{j=0}^{\pi-1} \delta_{f^j(p)}$. Note that this measure coincides with the Birkhoff limit of p, so that there is no ambiguity in the notation we employ.

We recall that, given a probability μ on M, its *push-forward* under f is the probability $f_*\mu$ defined by $f_*\mu(A) = \mu(f^{-1}(A))$ for every Borel measurable set $A \subset M$.

Finally, some new vocabulary regarding ergodically pathological (or well-behaved) dynamics:

Definition 3.4. *A dynamical system $f : M \to M$ is said to be*

w1) wonderful *if there exists a finite or countable family of physical measures μ_n such that*

$$m(\bigcup_n B(\mu_n)) = 1;$$

w2) wholesome *if m-a.e. point x has a well-defined Birkhoff limit μ_x;*

w3) weird *if m-a.e. point x has a well-defined Birkhoff limit μ_x, but f is totally singular and moreover admits no physical measure;*

w4) wacky *if m-a-e. point x does* not *have a well-defined Birkhoff limit; and*

w5) wicked *if f is not uniquely ergodic and moreover the averages*

$$m_n = \frac{1}{n}\sum_{k=1}^{n} f_*^k m$$

accumulate on the whole space of f-invariant measures: that is, if

$$\mathcal{M}_f(M) = \bigcap_{n\geq 0}\overline{\bigcup_{k\geq n} m_k}.$$

The five conditions above are set out in a roughly well-behaved-to-pathological order. Some implications between them are obvious: for example, every wonderful system is wholesome; a wacky system cannot be wholesome, nor can it admit any physical measures; and every weird system is wholesome but not wonderful. It is not obvious, but true, that every wicked system is wacky. In Section 4 we discuss further these conditions and the various implications between them.

Remark 3.5. We exclude uniquely ergodic systems from the wicked ones because otherwise every uniquely system would be both wicked and wonderful — which would sound great in a Lewis Carroll novel but impair our attempt to set the definitions in a roughly well-behaved-to-pathological order.

Theorem 3.6. *Let $\mathcal{C}$ denote either (i) the space $C^0(M)$ of all continuous maps from M to itself, where dim(M) is arbitrary, or (ii) the space* Homeo(M) *of all homeomorphisms from M to itself,* where M has dimension at leas 2 *(i.e., M is not the circle). Then there is a residual set $\mathcal{R} \subset \mathcal{C}$ such that every $f \in \mathcal{R}$ is weird.*

The behavior above shows that, from the point of view of view of Lebesgue measure, generic continuous maps (except for circle homeomorphisms) have very complicated global dynamics, but most individual orbits are quite well-behaved; this nicely parallels the conclusion of Hurley et al's book [AHK] regarding the behavior of orbits of Baire-residual points. In their words: "the dynamics of a generic homeomorphism is geometrically complicated but the behavior of most orbits is quite stable" (page 5).

The ergodic behavior of generic homeomorphisms of the circle is very different from the scenario of Theorem 3.6. Indeed, from an ergodic point of view, these are utterly well-behaved:

Theorem 3.7. *There is a residual set $\mathcal{R} \subset$* Homeo(S^1) *of all circle homeomorphisms such that every $f \in \mathcal{R}$ is wonderful. The set of physical measures μ_n of each $f \in \mathcal{R}$ is countably infinite and each physical measure is a periodic Dirac measure.*

Remark 3.8. The Dirac physical measures μ_n above enjoy a peculiar form of robustness: given any of the physical measures μ as described above and any $\varepsilon > 0$ there is an open neighborhood $\mathcal{U}$ of f in $\mathrm{Homeo}(S^1)$ such that if $g \in \mathcal{U}$ then g has a Dirac physical measure ν such that $d_H((supp(\mu), supp(\nu)) < \varepsilon$ and moreover $m(B(\mu) \triangle B(\mu^g)) < \varepsilon$. One is thus tempted to conclude that, when looked at individually, each physical measure of $f \in \mathcal{R}$ admits a weak* continuation. However, continuation is not the right word in this context, for two reasons. Firstly, for a generic element of $\mathrm{Homeo}(S^1)$, no physical measure is isolated, i.e. it is accumulated on by other physical measures, both in terms of its support and, consequenly (since they are all Dirac measures), in the weak* topology. The other reason is that even if $f \in \mathrm{Homeo}(S^1)$ has an isolated Dirac physical measure, e.g. as in the (non-generic) case of Morse-Smale diffeomorphisms, one can easily perturb f in the C^0 topology to obtain a new homeomorphism with an arbitrarily large number of physical measures near it.

The type of continuity that does hold, however, is that the weak* closure of the set of physical measures depends continuously on f in the Hausdorff topology in $\mathcal{M}_f(M)$ whenever $f \in \mathcal{R}$.

Although the ergodic properties of the generic systems studied in Theorems 3.6 and 3.7 are radically different from the global viewpoint, from the point of view of individual orbits they are quite similar in that they are wholesome — Lebesgue almost every orbit has a well defined Birkhoff limit. This is essentially due to an abundance, in both of these contexts, of *trapping regions*: open regions which are mapped strictly into their own interiors. We believe that the abundance of trapping regions is essentially equivalent to weirdness, and that in their absence wickedness will prevail.

Conjecture. *Let $\mathcal{C}$ be either (i) the complement in $C^0(M)$, with M of any dimension, of those maps that admits a trapping region, or (ii) the complement in* $\mathrm{Homeo}(M)$*, with M of dimension at least* 2*, of those homeomorphims that admits a trapping region. Then a generic element of $\mathcal{C}$ is wicked. (Note that, in either case, $\mathcal{C}$ is a Baire space since it is a closed subset of a complete metric space.)*

Our third and final result points in this direction, proving the conjecture to be true in the context of circle maps which are conjugated to expanding ones. More precisely, given an integer k with $|k| \geq 2$, let E_k denote the linearly induced expanding circle map of degree k, i.e. the map $x \mapsto kx$ mod 1.

Consider the sets

$$CE_k(S^1) = \{hE_kh^{-1} : h \in \text{Homeo}_+(S^1)\}.$$

Thus $CE_k(S^1)$ consists of all continuous circle maps which are topologically conjugated to E_k; hence to any C^1 map f of degree k, having an iterate f^n such that $|(f^n)'(x)| > 1$ for all $x \in S^1$ — the standard definition of expanding circle map. Likewise, the set

$$CE(S^1) = \bigcup_{|k|\geq 2} CE_k(S^1)$$

consists of all continuous circle maps topologically conjugate to some expanding map. It becomes a topological space by considering it as a subspace of $C^0(S^1)$. As such, it is neither closed nor open. It is a nowhere dense set which, by any reasonable standard, should be considered extremely meager. Still, by Proposition 7.1, $CE(S^1)$ is itself a Baire space, so it becomes relevant to ask what its generic properties are.

Theorem 3.9. *Generic elements of $CE(S^1)$ are wicked.*

So in this context generic dynamics is even more pathological: iteration of Lebesgue measure completely "deforms" it. In fact, a simpler argument than the one employed in the proof of Theorem 3.9 proves that, for $f \in \mathcal{R}$, the induced push-forward map f_* is transitive on $\mathcal{M}(M)$, having $\{f_*^n m : m \geq 0\}$ as a dense orbit. The proof of Theorem 3.9 is slightly more involved because it has to deal with the problem of showing that, when $f_*^n m$ gets near an invariant measure ν, it stays there long enough so that $\frac{1}{n}\sum_{k=0}^{n-1} f_*^k m$ gets near ν.

We end this section with a few remarks:

- One of the central themes of ergodic theory is of course entropy. We remark that by [Yan] every C^0-generic homeomorphism (except of course for circle homeomorphisms) and every C^0-generic continuous map has infinite topological entropy.

- Apart from the trivial case of Morse-Smale diffeomorphisms, very little about generic existence of physical measures is known in the C^1 topology. Campbell and Quas [CQ] used an approach based on thermodynamic formalism to prove that a generic C^1 expanding circle map has a unique physical measure. Their argument was recently adapted [Qiu] to C^1 generic hyperbolic attractors. It was also shown in [ABC]

that "tame" C^1-generic diffeomorphisms – which include transitive ones – exhibit a Baire-residual subset S of M such that the Birkhoff average μ_x exists for every $x \in S$: that is, C^1-generic tame diffeomorphisms are "Baire-wholesome". But as far as the authors know there are no results on the wholesomeness or wickedness of Lebesgue-a.e. point of the manifold in this context.

For higher regularity ($r > 1$), it is a classical result that Anosov diffeomorphisms, or more generally, Axiom A diffeomorphisms with no cycles, are open sets of wonderful maps. The same holds for uniformly expanding maps. Being open, these sets intersect any residual set. Efforts have been made to enlarge these sets to certain classes of diffeomorphisms admitting dominated splitting, by assuming non-uniform contraction or expansion in one of the invariant directions (instead of uniform contraction and expansion, which is the case for Axiom A systems). See [BV, ABV, And]. There is a result due to Tsujii [Tsu] which states that a C^r generic partially hyperbolic endomorphism on the 2-torus is wonderful, whenever $r \geq 19$. It is the most remarkable result in this direction in terms of technical sofistication.

- Another interesting question, raised independently by Ch. Bonatti and by E. R. Pujals in private discussions, is whether C^0-*densely* the dynamics is finitely wonderful (i.e., there is a finite set of physical measures the union of whose basins has full Lebesgue measure); this is a C^0-version of the Palis conjecture on finitude of attractors [Pal]. A partial (positive) answer to this question in the context of homeomorphisms is given by combining results of Moise, Shub, and Sinai-Ruelle-Bowen: the "C^0-Palis conjecture for homeomorphisms" holds in dimensions $d = 1, 2, 3$, where homeomorphisms can always smoothed by C^0-perturbations into diffeomorphisms (see [Moi], which in turn can (by [Shu]) be C^0-perturbed into structurally stable C^∞ diffeomorphisms, which are finitely wonderful by [Bow]). We note that in dimensions $d \geq 7$ there do exist non-smoothable homeomorphisms, by [Mil].

4 Wonderful, wholesome, weird, wacky, wicked

In this section we first discuss the "w" conditions defined in Definition 3.4 and the implications among them, and then point out some examples.

4.1 Implications

Some of the implications among the five w's are immediate or trivial and indeed have already been mentioned in the Introduction. They are:

- Every wonderful system is wholesome.
- Wacky system are not wholesome; moreover they admit no physical measures.
- Every weird system is wholesome but not wonderful.
- No weird system is wacky.

A less obvious implication is

Proposition 4.1. *Every wicked system is wacky.*

Proof. Suppose that f is not wacky. That is, that there exists a set A of positive Lebesgue measure such that for every $x \in A$, the Birkhoff limit $\mu_x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)}$ is well-defined. We shall prove that in this case, f is not wicked. If f is uniquely ergodic, there is nothing to prove. Suppose it is not; then there exist two distinct *ergodic* f-invariant measures ν_1, ν_2. To prove the proposition, it suffices to prove that $m_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m$ cannot accumulate on both ν_1 and ν_2.

Let m_A denote the normalized restriction of Lebesgue measure to A. Then the limit

$$\mu_A = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_A \tag{3}$$

is well-defined. In fact, the limit is the unique measure μ_A such that $\int \varphi d\mu_A = \int(\int \varphi d\mu_x) dm_A$ holds for every continuous $\varphi : M \to \mathbb{R}$. To see this, observe that

$$\lim_{n\to\infty} \int \varphi \; d\left(\frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_A\right) = \lim_{n\to\infty} \int \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(x)) \; dm_A(x) \tag{4}$$

$$= \int \left(\int \varphi \; d\mu_x\right) \; dm_A(x) = \int \varphi \; d\mu_A, \tag{5}$$

where the passage from (4) to (5) is justified by the dominated convergence theorem.

Note that if A has full Lebesgue measure, then we are done because then μ_A will be the only accumulation point of m_n. For the remainder of

the proof we therefore suppose that A^c has positive Lebesgue measure. We denote by m_{A^c} the normalized restriction of Lebesgue measure to A^c. To prove that m_n cannot accumulate on both ν_1 and ν_2, we start by fixing some $0 < \epsilon < m(A)/2$. Since ν_1 and ν_2 are mutually singular, there exists a continuous function $\varphi : M \to [0,1]$ such that $\int \varphi \, d\nu_1 < \epsilon$ and $\int \varphi \, d\nu_2 > 1-\epsilon$. Thus if m_n were to accumulate on both ν_1 and ν_2 we would have

$$\limsup_{n\to\infty} \int \varphi dm_n - \liminf_{n\to\infty} \int \varphi dm_n > 1 - 2\epsilon > \mu(A^c). \tag{6}$$

However, observing that

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k m = m(A^c)\frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_{A^c} + m(A)\frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_A, \tag{7}$$

we estimate

$$\limsup_{n\to\infty} \int \varphi dm_n \leq m(A^c) + m(A)\int \varphi d\mu_A \tag{8}$$

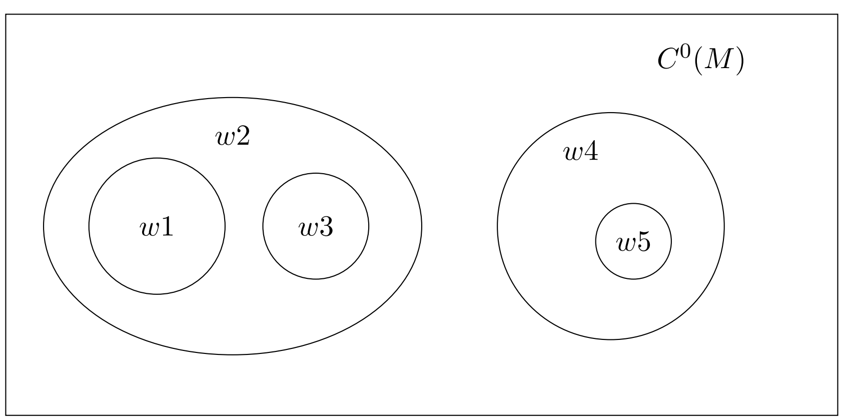
and

$$\liminf_{n\to\infty} \int \varphi dm_n \geq m(A)\int \varphi d\mu_A. \tag{9}$$

Therefore $\limsup_{n\to\infty} \int \varphi dm_n - \liminf_{n\to\infty} \int \varphi dm_n \leq m(A^c)$, contradicting (6).

□

In order to make the grand scheme of things clearer, we include an Euler diagram:



4.2 Examples

There are of course many well-known examples of systems which are wonderful (e.g, uniformly hyperbolic ones). We do not know of any example of weird systems in the literature. There are many examples of maps which have convergent Birkhoff averages Lebesgue almost everywhere and yet have no physical measures. The identity on any manifold or rational translations on tori are examples of such. We do not think these are weird at all and that is why we included the requirement of being totally singular into the definition of weird. Though as far as we the literature contains no examples of weird dynamics, Theorem 3.6 shows that weirdness is extremely abundant in the C^0 topology.

Wholesome systems which are neither wonderful nor weird may easily be concocted from the identity and the map $f(x) = \frac{1}{10}\sin^2(2\pi x)$ on S^1 — simply use the identity on the first half of the circle and f on the other.

Some authors [Mis, JT] study the notion of natural measures. It is usually defined as a measure μ such that, given any measure ν absolutely continuous with respect to Lebesgue measure, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \nu \to \mu. \tag{10}$$

Sometimes it is only required that (10) hold for all ν supported in the basin of a given topological attractor. If a continuous map has a unique physical measure whose basin is of full Lebesgue measure, then this measure is also natural. However, it was proved in [BB] that there are natural measures that are not physical. For some time it remained unclear whether it could be true that every ergodic natural measure is physical, until this was proved in [JT] not to be the case. Misiurewicz [Mis] gives an example of a continuous map $f : \mathbb{T}^2 \to \mathbb{T}^2$ having a natural measure, but with the pathological property that, for Lebesgue almost every $x \in \mathbb{T}^2$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}$ accumulates on the whole of $\mathcal{M}_f(M)$, which, in Misiurewicz's example, is a very rich set. In particular, his example proves that there are systems that are wicked but not wacky.

5 Proof of Theorem 3.6

Theorem 3.6 is a fairly straightforward consequence of the following Lemma:

Shredding Lemma. *Let $\mathcal{C}$ be be either the space $C^0(M)$ of all continuous maps from M to itself with dim$(M) \geq 1$ or the space* Homeo(M) *of all*

self-homeomorphisms of M with $\dim(M) \geq 2$. *Given any $\varepsilon > 0$, there is a dense subset $\mathcal{C}^{\varepsilon} \subset \mathcal{C}$ such that for every $f \in \mathcal{C}^{\epsilon}$ there is a family of pairwise disjoint open sets $U_1, \ldots, U_N$ such that*

i) each U_i is a trapping region: $f(\overline{U_j}) \subset U_j$ for every $j \in \{1, \ldots N\}$;

ii) each U_j has small Lebesgue measure:

$$m(U_j) < \varepsilon;$$

iii) the union of the sets U_j occupies, Lebesgue-wise, most of M:

$$m(\bigcup_{j=1}^{N} U_j) > 1 - \varepsilon;$$

iv) the sets U_j are "crushed" by iteration:

$$m(f(U_j)) < \epsilon \cdot m(U_j) \tag{11}$$

v) each U_j is strictly contained in the basin of a periodic cycle of sets with small diameter: there exist open sets $W_j^1, \ldots, W_j^{k_j}$ such that

(a) $\operatorname{diam}(W_j^i) < \epsilon$,

(b) $\overline{f^{k_j}(W_j^i)} \subset W_j^i$ *for every* $i \in \{1, \ldots, k_j\}$ *and*

(c)
$$\overline{U_j} \subset \bigcup_{n \geq 0} f^{-n}(W_j^1 \cup \ldots \cup W_j^{k_j}); \tag{12}$$

We first show how the Shredding Lemma implies Theorem 3.6, and later prove the Shredding Lemma itself.

Proof of Theorem 3.6. First note that all five conclusions of the shredding lemma are robust under small C^0 perturbations so that the sets $\mathcal{C}_{\epsilon}$ are, in fact, open and dense. Let $\mathcal{R}$ be the residual set obtained by intersecting a countable number of these:

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{U}_{\frac{1}{n}}.$$

We claim that every $f \in \mathcal{R}$ is weird. The easiest part is to prove that elements of $\mathcal{R}$ can have no physical measures. It follows from the observation that if $f \in \mathcal{U}_\epsilon$ then

$$\sup_{\mu \in \mathcal{M}_f} m(B(\mu)) < 2\epsilon. \tag{13}$$

Indeed, it follows from properties ii) and iii) of the Shredding Lemma that if there were a measure μ with $m(B(\mu)) \geq 2\epsilon$, then there would be points x and y of $B(\mu)$ which belong to different sets U_j and $U_{j'}$. But a bump function that takes the value 1 at U_j and 0 at $U_{j'}$ clearly shows that the points x, y cannot lie in the basin of the same measure, a contradiction.

To see that every $f \in \mathcal{R}$ is totally singular we make use of a little trick. Recall that $\mathcal{R}$ was obtained as a a countable intersection of sets $\mathcal{C}_{\frac{1}{n}}$. For each $n \in \mathbb{N}$ let V_n be the union of the trapping regions from the Shredding Lemma. Then $m(V_n) > 1 - 1/n$ and $m(f(V_n)) \leq 1/n$. Now consider

$$\Lambda = \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} V_{n^2} \tag{14}$$

Then Λ has full Lebesgue measure and $m(f(\Lambda)) \leq \sum_{n=k}^{\infty} \frac{1}{n^2}$ for every $k \in \mathbb{N}$, by the crushing property (11). Hence $f(\Lambda) = 0$. The set Λ^c has total Lebesgue measure but its pre-image under f has zero Lebesgue measure. Hence f is totally singular.

It remains to show that elements of $\mathcal{R}$ have convergent Birkhoff averages Lebesgue almost everywhere. Thus we fix $f \in \mathcal{R}$ and consider the set

$$\Lambda = \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} V_n. \tag{15}$$

(We could still work with $\Lambda = \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} V_{n^2}$ but for the current purpose it is not a very natural choice as we have no longer any need for summabillity.) Fix a continuous function $\varphi : M \to \mathbb{R}$ and consider some point $x \in \Lambda$. We will show that, for every $\delta > 0$,

$$\limsup_n \frac{1}{n} \sum_{j=1}^{n} \varphi(f^j(x)) - \liminf_n \frac{1}{n} \sum_{j=1}^{n} \varphi(f^j(x)) \leq 2\delta. \tag{16}$$

Once that is done, the proof is complete.

Since φ is uniformly continuous, for large enough $n_0 \in \mathbb{N}$ it holds that $|\varphi(x) - \varphi(y)| < \delta$ whenever $d(x, y) < \frac{1}{n_0}$. Now, by the definition of Λ, there is some iterate $f^n(x)$ of x belonging to the some set W_j^i with diameter smaller

than $\frac{1}{n_0}$ and such that $f^{k_j}(\overline{W_j^i}) \subset W_j^i$ for some $k_j \geq 1$. Since the veracity of (16) does not change if x is replaced with some iterate of itself, we might assume that $x \in W_j^{k_i}$ for simplicity. Observe that $d(f^{\ell \cdot k_j + r}(x), f^r(x)) < 1/n_0$ and, consequently, $|\varphi(f^{\ell \cdot k_j + r}(x)) - \varphi(f^r(x))| < \delta$ for every $\ell \in \mathbb{Z}$. Thus, writing an arbitrary integer $n \geq 0$ as $n = \ell \cdot k_j + r$ with $0 \leq r < k_j$ and $\Gamma = \frac{1}{k_j} \sum_{m=0}^{k_j - 1} f^m(x)$ we obtain the estimate

$$\Gamma - \delta - \frac{r}{n} \cdot \|\varphi\|_{C^0} < \frac{1}{n} \sum_{j=1}^{n} \varphi(f^j(x)) < \Gamma + \delta + \frac{r}{n} \cdot \|\varphi\|_{C^0}, \tag{17}$$

of which (16) is a direct consequence. □

We prove the Shredding Lemma for homeomorphisms and continuous mappings separately, beginning with the latter (and much easier) case. Our proof relies on the existence of triangulations of smooth manifolds. There is probably nothing fundamental about this, and we find it likely that Theorem 3.6 is generalizable to non-smoothable topological manifolds that do not admit triangulations (in which case the role of Lebesgue measure could be represented by any non-atomic measure positive on open sets). However, the decomposition of the manifold into simplices provides a very handy set of coordinates defined on convex subsets of $\mathbb{R}^n$ in which we can perform the explicit perturbations used in the proof.

Thus by a "*triangulation*" we simply mean a finite collection $\mathcal{R} = \{R_1, \ldots, R_r\}$ of compact subsets of M homeomorphic to (say) the simplex

$$\Delta_d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } i\}, \tag{18}$$

together with homeomorphisms $\xi_i : R_i \to \Delta_d$, such that $m(R_i) = 0$ for every i and such that if $i \neq j$ then i) either $R_i \cap R_j = \emptyset$ or ii) $R_i \cap R_j \subset \partial R_i$. Cairns triangulation [Cai] does the job. The diameter of a subset $A \subset M$ is defined as $\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y)$ and the diameter of a triangulation $\mathcal{R} = \{R_1, \ldots, R_r\}$ is $\operatorname{diam} \mathcal{R} = \max\{\operatorname{diam}(R_1), \ldots, \operatorname{diam}(R_r)\}$. We shall say loosely that a triangulation is *fine* if its diameter is small. By dividing the standard simplex into smaller simplices we may subdivide any triangulation into a finer one. In particular, there are arbitrarily fine triangulations of M.

Proof of the Shredding Lemma for continuous mappings. Fix some $f_0 \in C^0(M)$ and $\epsilon > 0$. We shall describe how to find $f \in C^0(M)$ with $d(f, f_0) < \epsilon$ such that f satisfies items i) to v) of the Shredding Lemma.

Let $\mathcal{R} = \{R_1, \ldots, R_r\}$ be a fine triangulation of M. Precisely how fine it should be is a question we postpone to the end of the proof. Subdivide each simplex $R_i \in \mathcal{R}$ into a union of subsimplices $R_i = R_i^1 \cup \ldots \cup R_i^s$ so that

$$m\left(\bigcup_{i=1}^{r} R_i^j\right) < \epsilon \quad \text{for every } 1 \leq j \leq s. \tag{19}$$

For $\delta > 0$, let $\mathrm{Int}_\delta(R_i^j) = \{x \in R_i^j : d(x, \partial R_i^j) > \delta\}$. We choose δ small enough so that

$$m\left(\bigcup_{i=1}^{r}\bigcup_{j=1}^{s} \mathrm{Int}_\delta(R_i^j)\right) > 1 - \epsilon. \tag{20}$$

For each $i \in \{1, \ldots, r\}$ choose some $\tau(i) \in \{1, \ldots, r\}$ such that $f(R_i) \cap R_{\tau(i)} \neq \emptyset$. Choose points p_i^j in $\mathrm{Int}_\delta(R_i^j)$. We shall define f in such a way that $f = f_0$ on each ∂R_i^j and such that f takes the constant value p_i^j on the whole of $\overline{\mathrm{Int}_\delta(R_i^j)}$. That is,

$$f|\partial R_i^j = f_0|\partial R_i^j \quad \text{and} \tag{21}$$

$$f|\,\mathrm{Int}_\delta(R_i^j) = p_{\tau(i)}^j \tag{22}$$

for every $1 \leq i \leq r,\ 1 \leq j \leq s$.

It is clear that any $f : M \to M$ satisfying (21) and (22) also satisfies itens i) to v) of the Shredding Lemma with $U_j = \bigcup_{i=1}^{r} \mathrm{Int}_\delta(R_i^j)$. Hence all we have to do now is to make sure that f can be extended continuously to M in a way that $d(f, f_0) < \epsilon$. By supposing that the triangulation $\mathcal{R}$ is sufficiently fine, we may assume that the charts ξ_i extend to homeomorphisms $\psi_i : U_i \to \psi_i(U_i) \subset \mathbb{R}^n$ on open sets $U_i \supset R_i$ are such that, for every $1 \leq i \leq r$, the closed convex hull of $f_0(R_i) \cup R_{\tau(i)}$ is contained in $U_{\tau(i)}$. More precisely, writing $i' = \tau(i)$, we require that $f_0(R_i) \subset U_{i'}$ and that the closed convex hull of $\psi_{i'}(f_0(R_i) \cup R_{i'})$ be contained in $\psi_{i'}(U_{i'})$. In the linear structure induced by ψ_i and $\psi_{i'}$ we may define f on each $R_i^j,\ j = 1, \ldots, s$ explicitly by fixing a continuous function $\varphi_i^j : R_i^j \to [0, 1]$ satisfying $\varphi_i^j|\partial R_i^j \equiv 0$ and $\varphi_i^j|\,\mathrm{Int}_\delta(R_i^j) \equiv 1$ and define f by

$$\psi_{i'} \circ f \circ \psi_i^{-1}(x) = (1 - \varphi_i^j(x)) \cdot \psi_{i'} \circ f_0 \circ \psi_i^{-1}(x) + \varphi_i^j(x) p_{i'}^j \tag{23}$$

for every $x \in R_i^j$. This is well-defined since $f|R_i^j$ belongs to the closed convex hull of $f_0(R_i) \cup R_{\tau(i)}$. The C^0 distance between f_0 and f depends on the diameter of $\mathcal{R}$ as well as the constants of uniform continuity of f_0, ψ_i and

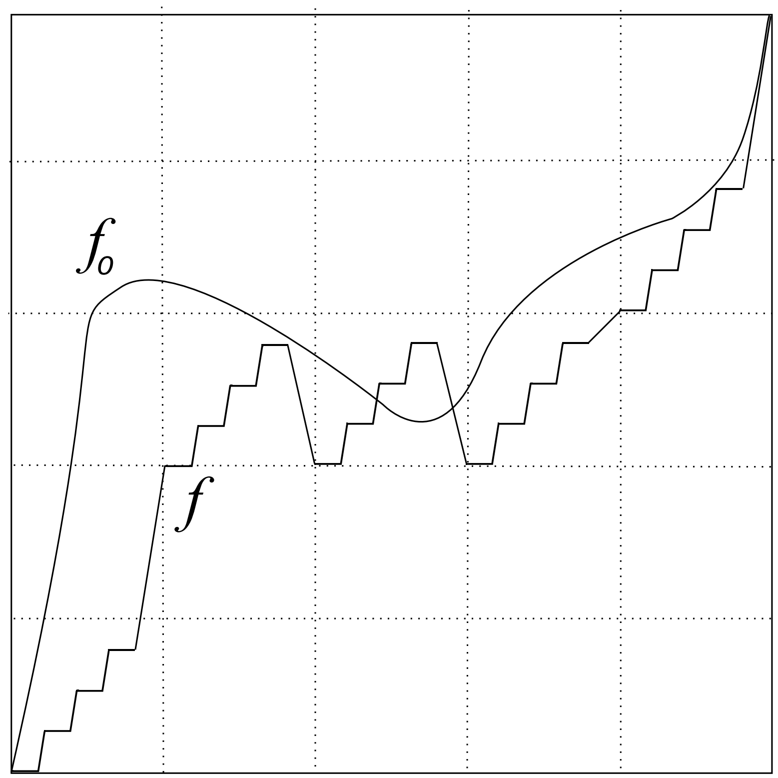


Figure 1: On S^1, the Shredding Lemma is obtained by perturbing the original map with a step function. In the above illustration, there are four trapping regions, each of which is a union of five tiny intervals.

$\psi_{i'}^{-1}$. In case we do not have $d(f, f_0) < \epsilon$ we refine our tentative triangulation $\mathcal{R}$ by subdividing each of its elements into a large number of sub-simplicies, each of which is defined using the chart ψ_i, and repeat the argument with this finer triangulation. That way, the constants of uniform continuity are maintained so that $d(f, f_0) < \epsilon$ provided that the new triangulation is fine enough.

$\square$

Proof of the Shredding Lemma for homeomorphisms in higher dimensions. We fix an arbitrary $f_0 \in \mathrm{Homeo}(M)$ and $\epsilon > 0$. Consider a triangulation $\mathcal{R} = \{R_1, \ldots, R_r\}$ of M with diameter $\epsilon_0 < \epsilon$. We will define a homeomorphism $h \in \mathrm{Homeo}(M)$ satisfying $h(R_i) = R_i$ for every $1 \leq i \leq r$ (hence $d(h, \mathrm{id}) < \epsilon_0$) such that $f = f_0 \circ h$ satisfies properties i) to v) in the Shred-

ding Lemma. Then, by uniform continuity of f_0, $d(f, f_0) < \epsilon$ provided that ϵ_0 is sufficiently small.

For each $i \in \{1, \dots, r\}$ we choose some $\tau(i) \in \{1, \dots, r\}$ such that $\mathrm{Int}(f(R_i)) \cap \mathrm{Int}(R_{\tau(i)}) \neq \emptyset$. Once such a map $\tau : \{1, \dots, r\} \circlearrowleft$ is chosen we pick, for each $i \in \{1, \dots, r\}$, some $p_i \in \mathrm{Int}(R_i \cap f^{-1}R_{\tau(i)})$. Using the linear structure from $\mathbb{R}^n$ we now write down explicit formulae defining homeomorphisms $\varphi_i : M \to M$, whose restriction to $M \setminus \mathrm{Int}\, R_i$ is the identity. The important property of φ_i is that it sends points points in the interior of R_i towards p_i. More precisely we want φ_i to satisfy

$$\varphi_i(\overline{\mathrm{Int}_\delta(R_i)}) \subset B_\delta(p_i), \tag{24}$$

for some small $\delta > 0$ where, again, $\mathrm{Int}_\delta(R_i)$ denotes the set of points in R_i whose distance to ∂R_i is larger than δ. We take as an explicit choice

$$\varphi_i(x) = \begin{cases} x & \text{if } x \notin \mathrm{Int}\, R_i \\ \alpha(x)^T x + (1 - \alpha(x)^T)p_i & \text{if } x \in \mathrm{Int}\, R_i, \end{cases} \tag{25}$$

where

$$\alpha(x) = \frac{d(x, p_i)}{d(x, p_i) + d(x, \partial R_i)}. \tag{26}$$

One verifies that, no matter how small we take $\delta > 0$, we may always choose $T > 0$ sufficiently large to obtain (24). We choose δ small enough that $f_0(\overline{B_\delta(p_i)}) \subset \mathrm{Int}_\delta(R_{\tau(i)})$ for every $i \in \{1, \dots, r\}$. Increasing T further, if necessary, guarantees that

$$\varphi_i(\overline{\mathrm{Int}_\delta(R_i)}) \subset f_0^{-1}(\mathrm{Int}_\delta(R_{\tau(i)})). \tag{27}$$

Hence $f_0 \circ \varphi_1 \circ \dots \circ \varphi_r$ maps $\overline{\mathrm{Int}_\delta(R_i)}$ into $B_\delta(p_{\tau(i)}) \subset \mathrm{Int}_\delta(R_{\tau(i)})$ for each $i \in \{1, \dots, r\}$.

Since the map $\tau : \{1, \dots, r\} \to \{1, \dots, r\}$ acts on a finite space, each point must be pre-periodic. We may therefore write $\{1, \dots, r\}$ as a union $I_1 \cup \dots \cup I_s$ of pairwise disjoint subsets, each containing exactly one periodic orbit and all its pre-images under τ. Already at this point, we can prove items i) iii) and iv) of the Shredding Lemma; these are the essential ingredients used to obtain the singularity property of generic homeomorphisms. For that, all we have to do is to take $\delta > 0$ small and choose $T > 0$ large enough that $f = f_0 \circ \varphi_1 \circ \dots \circ \varphi_r$ maps $\overline{\mathrm{Int}_\delta(R_i)}$ into $B_\delta(f_0(p_i))$ for each $i \in \{1, \dots, r\}$, as commented above. Then f satisfies items i), iii) and iv) with

$$U_k = \bigcup_{i \in I_k} \mathrm{Int}_\delta(R_i), \quad k \in \{1, \dots s\}. \tag{28}$$

The perturbation required to obtain properties ii) and v) of the Shredding Lemma, the part responsible for non-existence of physical measures as well and Lebesgue almost everywhare convergence of Birkhoff averages, is much more nettlesome. It requires us to slice the above trapping regions U_k from (28) into smaller ones.

To this end, let $P = \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$ be a set containing exactly one point of each periodic orbit under τ, labelled conveniently so that $i_k \in I_k$ for every $k = 1, \ldots s$. Write π_{i_k} for the period of i_k. For $i \in P$ we will slice R_i in a pizza-like fashion (see figure 5) and redefine φ_i so that some of the slices form new trapping regions. This pizza slice construction is not entirely arbitrarily. To perform it, we have to assume that $\dim M \geq 2$. It is indeed inevitable that this assumption enters the proof at some point, since otherwise we would contradict Theorem 3.7.

Pizza Slize

A convenient way to slice R_i in a pizza-like fashion is to pick a tiny $\dim M-1$ dimensional sphere $\mathbb{S}_i$ centered around p_i, contained in $\operatorname{Int} R_i$ with $f(\mathbb{S}_i) \subset \operatorname{Int} R_{\tau(i)}$. In the triangulation coordinates of R_i, $\mathbb{S}_i = \{x \in R_i : \|x - p_i\| = \rho_i\}$ for some $\rho_i > 0$. Let $\mathcal{Q}_i = \{Q_i^1, \ldots, Q_i^{q_i}\}$ be a fine triangulation of $\mathbb{S}_i$ and, for each Q_i^j, $j = 1, \ldots, q_i$, let S_i^j be the union of all line segments from p_i to ∂R_i that intersect Q_i^j. We refer to the collection $\mathcal{S}_i = \{S_i^1, \ldots, S_i^{q_i}\}$ as a *slicing* of R_i. Each S_i^j will then be a closed convex set. By taking $\operatorname{diam} \mathcal{Q}$ small, we make sure that the volume of each slice $S_i^j \in \mathcal{S}_i$ is as small as we want.

For each $j \in \{1, \ldots, q_i\}$ we choose some $\gamma_i(j) \in \{1, \ldots, q_i\}$ such that $\operatorname{Int} S_i^j \cap \mathbb{S}_i \cap f^{-\pi_i}(\operatorname{Int} S_i^{\gamma_i(j)}) \neq \emptyset$, and points $s_i^j \in \operatorname{Int} S_i^j \cap \mathbb{S}_i \cap f^{-\pi_i}(\operatorname{Int} S_i^{\gamma_i(j)})$. (Recall that π_i is the period of i under τ.) We perturb the identity inside each S_i^j so that most points of S_i^j are mapped to a small neighbourhood of s_i^j:

$$\phi_i(x) = \begin{cases} x & \text{if } x \notin \operatorname{Int} R_i \\ \beta_i^j(x)^T x + (1 - \beta_i^j(x)^T) s_i^j \text{ if } z \in S_i^j & \text{if } x \in S_i^j, \end{cases} \tag{29}$$

where

$$\beta_i^j(x) = \frac{d(x, s_i^j)}{d(x, s_i^j) + d(x, \partial S_i^j)}. \tag{30}$$

Again, ϕ_i leaves each S_i^j invariant due to convexity. The same is therefore true about $\phi = \phi_r \circ \ldots \circ \phi_1$. Moreover, given any $\delta > 0$, by taking $T >$

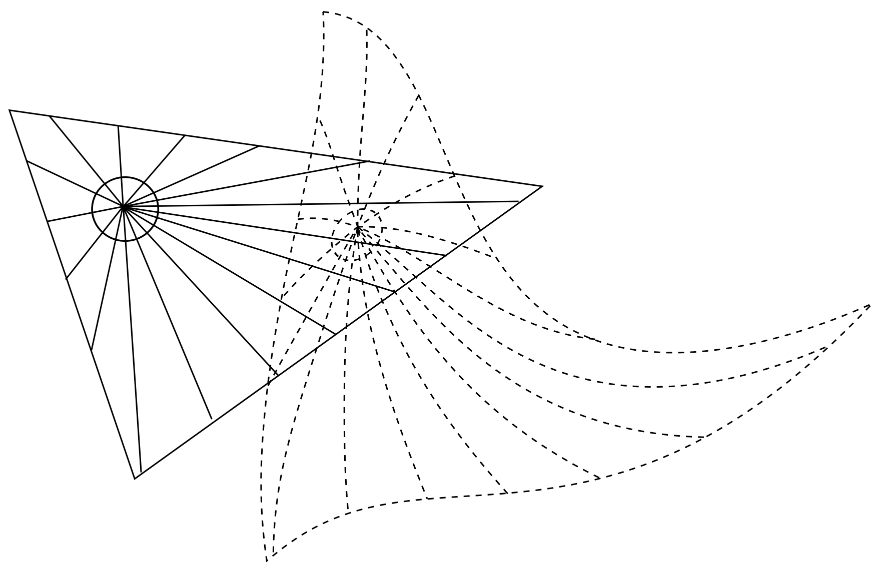


Figure 2: Pizza-Slice decomposition of R_i and $f^{\pi_i}(R_i)$.

0 sufficiently large, we ensure that $(f_0 \circ \phi)^{\pi_i}$ leaves $\bigcup_{j=1}^{q_i} \mathrm{Int}_\delta(S_i^j)$ strictly invariant. More precisely, $(f_0 \circ \phi)^{\pi_i}(\overline{\mathrm{Int}_\delta(S_i^j)}) \subset \mathrm{Int}_\delta(S_i^{\gamma_i(j)})$ for every $j \in \{1, \ldots, q_i\}$.

Tunelling

The whole point of slicing each R_i into $S_i^1 \cup \ldots \cup S_i^{q_i}$ is to decompose the trapping region (28) into smaller ones. But, depending on the dynamics of γ_i, our effort so far may be fruitless. Suppose, for example, that γ_i sends every $j \in \{1, \ldots, q_i\}$ to the same point, k say. Then every point in $\bigcup_{j=1}^{q_i} \mathrm{Int}_\delta(S_i^j)$ will fall into S_i^k under $(f_0 \circ \phi)^{\pi_i}$. A similar problem appears if γ acts as a cyclic permutation on $\{1, \ldots, q_i\}$, i.e. if the whole of $\{1, \ldots, q_i\}$ is a periodic orbit under γ. Then clearly $\bigcup_{j=1}^{q_i} \mathrm{Int}_\delta(S_i^j)$ is an indecomposable trapping region of $(f_0 \circ \phi)^{\pi_i}$ (i.e. not a union of smaller trapping regions). To deal with these issues we will define perturbations, baptized *tunellings*, that transfer mass between the s_i^j, effectively changing the dynamics of γ_i. After doing so, the new dynamics, $\tilde{\gamma}_i$, will have $\sigma_i = \# \mathrm{Im}\, \gamma_i$ periodic orbits. Moreover, the basins on the perturbed dynamics on M will all be of comparable size.

To this end we create a partition $\mathcal{P}_i = \{P_i^1, \ldots, P_i^{\sigma_i}\}$ of $\{1, \ldots, q_i\}$

such that $\ell \in P_i^\ell$ for every $\ell \in \operatorname{Im}\gamma_i$ and

$$|m\left(\bigcup_{j\in P_i^k} S_i^j\right) - m\left(\bigcup_{j\in P_i^\ell} S_i^j\right)| < \max_{j\in\{1,\dots,q_i\}} m(S_i^j) \tag{31}$$

for every $k, \ell \in \{1, \dots, \sigma_i\}$. It is crucial that the number $\sigma_i = \#\operatorname{Im}\gamma_i$ can be taken as large as desired by taking $\mathcal{Q}_i$ sufficiently fine. That way, property (31) implies that

$$\lim_{\operatorname{diam}\mathcal{Q}_i\to 0}\left\{\max_{k=1,\dots,\sigma_i} m\left(\bigcup_{j\in P_i^k} S_i^j\right)\right\} = 0 \tag{32}$$

For each pair $k, \ell \in \{1, \dots, q_i\}$ we denote by $C_i^{k,\ell}$ the closed convex hull of $B_\delta(s_i^k) \cup B_\delta(s_i^\ell)$. The geometry of the sphere is such that if δ is sufficiently small, then $B_\delta(s_i^t) \cap C_i^{k,\ell}$ for every $t \neq k, \ell$. Now, for each $\ell \in \{1, \dots, \sigma_i\}$ and each $k \in P_i^\ell$ we define a tunneling map

$$\psi_i^{k,\ell}(x) = \begin{cases} x & \text{if } x \notin \operatorname{Int} C_i^{k,\ell} \\ \beta(x)^T x + (1-\beta(x)^T)s_i^\ell & \text{if } x \in C_i^{k,\ell}. \end{cases} \tag{33}$$

The effect of $\psi_i^{k,\ell}$ is to transfer nearly all the mass in $B_\delta(s_i^k)$ into $B_\delta(s_i^\ell)$.

For each $\ell \in \{1, \dots, \sigma_i\}$ we write $P_i^\ell = \{k_i^1, \dots, k_i^\ell\}$ and define

$$\psi_i^\ell = \psi_i^{k_i^\ell,\ell} \circ \dots \circ \psi_i^{k_i^1,\ell}, \tag{34}$$

$$\varphi_i = \psi_i^m \circ \dots \circ \psi_i^1 \circ \phi_i, \tag{35}$$

$$f = f_0 \circ \varphi_r \circ \dots \circ \varphi_1. \tag{36}$$

Furthermore, let $W_{i,j}^\ell = f^{-\ell+1} S_i^j$ for $\ell \in \{1, \dots, \pi_i\}$.

We have $f^{\pi_i}(\overline{W_{i,j}^\ell}) \subset W_{i,j}^\ell$ for every $i \in \{1, \dots, r\}$, $j \in \{1, \dots, q_i\}$ and $\ell \in \{1, \dots \pi_i\}$. Moreover, taking

$$U_{i,j} = \left(\bigcup_{n=0}^{r} f^{-n} S_i^j\right) \cap \left(\bigcup_{i=0}^{r} \operatorname{Int}_\delta(R_i)\right) \tag{37}$$

we have $\overline{U}_{i,j} \subset \bigcup_{n=0}^\infty f^{-n}(W_{i,j}^1 \cup \dots \cup W_{i,j}^{\pi_i})$. By construction, f satisfies the Shredding Lemma, with $U_{i,j}$ playing the roles of the U_j and the $W_{i,j}^\ell$ playing the roles of the W_j^i as long as

1. the diameter of $\mathcal{R}$ is small enough,
2. the diameter of each each $\mathcal{Q}_i$ is small enough,
3. the number $\delta > 0$ is small enough, and
4. the number $T > 0$ is large enough.

□

6 Proof of Theorem 3.7

We shall prove that the nonwandering sets of generic homeomorphisms in $H(S^1)$ have zero Lebesgue measure and are Cantor sets of periodic points, and then deduce the desired dynamical consequences from this. We remark that Akin-Hurley-Kennedy have already proved (see Theorem 6.4 in page 68 of [AHK]) that the nonwandering sets of generic homeomorphisms of arbitrary manifolds are Cantor sets, but we include a full proof i) because in the circle the proof is simpler and shorter; ii) for the sake of completeness; and iii) because we use some of the vocabulary of the proof when we subsequently prove that these nonwandering sets have zero Lebesgue measure.

Proposition 6.1. *The nonwandering sets of generic homeomorphisms in* $\mathrm{Homeo}(S^1)$ *are zero Lebesgue measure Cantor sets of periodic points.*

Proof. We deal only with the case of orientation-preserving homeomorphisms, as the orientation-reversing case may be obtained through minor modifications. The proposition is proved in a series of steps:

Step 1: There is an open-and-dense subset $\mathcal{O}$ of $\mathrm{Homeo}(S^1)$ such that every $f \in \mathcal{O}$ has rational rotation number; moreover given any $f \in \mathcal{O}$ then its rotation number is constant in a neighborhood of f.

First, by closing recurrent orbits it follows that there is a dense subset $\mathcal{D}$ of $\mathrm{Homeo}(S^1)$ such that every $f \in \mathcal{D}$ has at least one periodic orbit (i.e., a rational rotation number). With an additional small C^0 perturbation one produces a topologically transversal periodic orbit p (i.e., a periodic point of period say π such that that f^π has a lift $F : \mathbf{R} \to \mathbf{R}$ satisfying $F(x) < F(p) = p < F(y)$ or $F(x) > F(p) = p > F(y)$ for some $x < p < y$). The existence [1] of a transversal periodic orbit is a C^0-open condition by the intermediate value theorem; this

[1]but not the uniqueness

implies that there is an open-and-dense subset $\mathcal{O}$ of Homeo(S^1) such that every $f \in \mathcal{O}$ has rational rotation number; and moreover by construction given any $f \in \mathcal{O}$ then its rotation number is constant in a neighborhood of f. (Remark: It is well-known that if the rotation number is rational then $\Omega(f) = Per(f)$.)

Step 2: There is a residual subset $\mathcal{R}_1$ of $\mathcal{O}$ (and hence of Homeo(S^1)) such that the nonwandering set of every $f \in \mathcal{R}_1$ has empty interior.

Fix a countable open basis $\{I_k\}$ of intervals of the circle. Given an interval I_k of the basis and a homeomorphism $f \in \mathcal{O}$ whose periodic orbits have a given period π, we can always perturb f so that $f^\pi|_{I_k}$ does not coincide with the identity – which means that the set I_k does not consist of periodic points; clearly this is a C^0-open condition. So there is an open-and-dense set $\mathcal{A}_k$ of homeomorphisms f such that $I_k \setminus Per(f)$ is (open and) nonempty. Taking the intersection we obtain a set

$$\mathcal{R}_1 \equiv \bigcap_{k \in \mathbb{N}} \mathcal{A}_k$$

which is residual in $\mathcal{O}$ and such that by construction if $f \in \mathcal{R}_1$ then $Per(f)$ does not contain any interval.

Step 3: There is a residual subset $\mathcal{R}_2$ of $\mathcal{O}$ (and hence of $H(S^1)$) such that the nonwandering set of every $f \in \mathcal{R}_2$ is perfect.

Given $k, n \in \mathbb{N}$ we shall prove the following: there is an open-and-dense subset $\mathcal{B}_n^k$ of $\mathcal{O}$ such that if $f \in \mathcal{B}_n^k$ is such that $Per(f) \cap I_k \neq \emptyset$, then $\#(Per(f) \cap I_k) \geq n$. Indeed, given $f \in \mathcal{O}$, then either (i) $Per(f) \cap I_k = \emptyset$ – and this is a C^0-open condition by the upper-semicontinuous variation of the set $Per(f)$ with f –; or (ii) $Per(f) \cap I_k \neq \emptyset$. In the second case, we perturb f around (the preimage of) some periodic point in $Per(f) \cap I_k$ so as to "unfold it" into at least n topologically transverse periodic points $p_1, \ldots, p_n \in I_k$; this condition is also C^0-open. This yields the desired set $\mathcal{B}_n^k$. Now take the (residual) intersection $\mathcal{R}_2 \equiv \bigcap_{n,k \in \mathbb{N}} \mathcal{B}_n^k$; by construction, given $f \in \mathcal{R}_2$ and any basic interval I_k then either $Per(f) \cap I_k = \emptyset$ or $card(Per(f) \cap I_k) = \infty$. This clearly implies that the set $Per(f)$ (which coincides with $\Omega(f)$) is perfect.

The homeomorphisms in $\mathcal{R}_1 \cap \mathcal{R}_2$ have Cantor nonwandering sets which consist of periodic orbits: their nonwandering sets are (i) of course compact, (ii) coincide with the set of periodic points by rationality of the rotation

number, (iii) have empty interior by step 2, and (iv) are perfect by step 3. It remains to show that generically the set of periodic points has zero Lebesgue measure.

- Step 4: There is a residual subset $\mathcal{R}_3$ of $\mathcal{O}$ (and hence of $H(S^1)$) such that the set of periodic points $Per(f)$ of every $f \in \mathcal{R}_3$ has zero Lebesgue measure.

  Given any homeomorphism $f \in \mathcal{O}$, we can via a small C^0-perturbation smooth it into a diffeomorphism $f \in \mathcal{O}$, which in turn can be perturbed into a Morse-Smale diffeomorphism, which has a finite (and hence zero Lebesgue-measure) set of periodic points. That is, there is a dense subset $\mathcal{D}$ of $\mathcal{O}$ which consists of homeomorphisms f such that $m(Per(f)) = 0$. Now, given $\varepsilon > 0$, by the upper semicontinuity of the map $f \to Per(f)$ there is an open neighborhood $\mathcal{U}_\varepsilon$ of f in $\mathcal{O}$ such that if $g \in \mathcal{U}_\varepsilon$ then $m(Per(g)) < \varepsilon$. Define now $\mathcal{W}_\varepsilon \equiv \bigcup_{f\in\mathcal{D}} \mathcal{U}_\varepsilon$ and set $\mathcal{R}_3 \equiv \bigcap_{n\in\mathbb{N}} \mathcal{W}_{\frac{1}{n}}$ to obtain the desired residual subset of $\mathcal{O}$.

$\square$

We now deduce Theorem 3.7 from Proposition 6.1.

Proof of Theorem 3.7. Again, we only deal explicity with the case of orientation preserving homeomorphisms, leaving the details of the orientation reversing case to the reader. Thus let f be a generic homeomorphism in $\text{Homeo}_+(S^1)$ whose periodic points have period π. Then the (open and full-Lebesgue) set $S^1 \setminus Per(f)$ consists of a countable union of pairwise disjoint open intervals I such that $f^\pi(I) = I$ and moreover the extremes of I are two periodic points p_1 (on the left) and p_2 (on the right), which are necessarily extremal points of the Cantor set $Per(f)$.

Let F be the lift of f^π to the real line which has fixed points. Given an interval I as above, there are two possible cases: either the graph of F restricted to I is below the identity, or else the graph of F restricted to I is above the identity.

In the first case, by dynamical monotonicity all of the points $x \in I$ converge in the future to the orbit of p_1: $d(f^k(x), f^k(p_1)) \to 0$; in the second case, the points x of I converge in the future to the orbit of p_2: $d(f^k(x), f^k(p_2)) \to 0$. This means that the periodic Dirac measure associated to the orbit of p_1 (in the first case) or of p_2 (in the second case) contains I in its basin of attraction. In other words, Lebesgue-a.e. point of the circle belongs to the basin of attraction of the Dirac measure associated

to an extremal point of the Cantor set $Per(f)$. This shows that f is indeed countably wonderful, as claimed. $\square$

7 Proof of Theorem 3.9

We recall the statement of Theorem 3.9: A generic continuous circle map, topologically conjugated to a linear expanding one, is wicked. 'Generic' here means 'generic in the induced C^0 topology on the set of all continuous maps conjugated to a linear expanding one'. In proving the theorem, we do not work directly with the maps themselves, but rather with the conjugating homeomorphisms. More precisely, what we actually prove is the following: Let E denote any linear expanding circle map. Given a generic circle homeomorphism h, the map $f = h^{-1}Eh$ is wicked. The statement of Theorem 3.9 then follows by the following proposition.

Proposition 7.1.

1. *The decomposition $CE(S^1) = \bigcup_{|\ell|\geq 2} CE_\ell(S^1)$ is a decomposition into isolated sets.*
2. *For each integer ℓ with $|\ell| \geq 2$, $CE_\ell(S^1)$ is locally homeomorphic to $\mathrm{Homeo}_+(S^1)$. In fact, the map*

$$\mathrm{Homeo}_+(S^1) \to CE_\ell(S^1) \tag{38}$$

$$h \mapsto h^{-1}E_\ell h \tag{39}$$

is a $|\ell - 1|$-to-one surjection, mapping a neighbourhood of every $h \in \mathrm{Homeo}_+(S^1)$ homeomorphically onto its image. In particular, $CE(S^1)$ is a Baire space.

Proof. Suppose that f and g are two elements of $CE_\ell(S^1)$ of degrees ℓ and m, say. Let F and G be lifts of f and g respectively. If $\ell \neq m$, then $deg(f - g) = \ell - m \neq 0$. It other words, $\{F(1) - G(1)\} - \{F(0) - G(0)\}$ is a nonzero integer. By the mean value theorem, there exists $0 \leq x_0 \leq 1$ such that $F(x_0) - G(x_0) = n + 1/2$ for some integer n. Consequently $d^0_C(f, g) = 1/2$. This proves item 1 of the theorem.

To prove item 2 we fix some ℓ with $|\ell| \geq 2$ and write $E = E_\ell$, to simplify the notation a bit. We also fix some $h \in \mathrm{Homeo}_+(S^1)$ and consider the map $f = hEh^{-1}$, which belongs to $CE_\ell(S^1)$. Since f is conjugated to E, it must have exactly $|\ell - 1|$ fixed points, say $p_1, \ldots, p_{|\ell-1|}$, and one of these must be mapped by h to the poitn 0 in $\mathbb{R}/\mathbb{Z}$. Once specified, the images, under

h, of all other fixed points are also specified, since they must be mapped to the remaining fixed points of E in a particular order. So are the images of all the fixed points of f^2, since they must be mapped into the set of fixed points of E^2 respecting a given order. By the same reasoning, the images of all periodic points are determined. It is thus clear by the density of periodic points that to specify which p_i is mapped to the origin really determines h. It follows that, given $f \in CE_\ell(S^1)$, there are *at most* $|\ell - 1|$ choices of $h \in \mathrm{Homeo}_+(S^1)$ such that $f = h^{-1}E_\ell h$. On the other hand, composing h on the left by a rotation R whose angle is a multiple of $(\ell - 1)^{-1}$, we obtain a new homeomorphism $h' = Rh$ such that $(h')^{-1}E_\ell h' = f$. Hence there are *at least* $|\ell - 1|$ choices of homeomorphisms that conjugate f to E_ℓ. We have therefore shown that there are *precisely* $|\ell - 1|$ homeomorphisms conjugating a given f to E_ℓ, and that they all differ by left composition of a rigid rotation of angles that are multiples of $|\ell - 1|^{-1}$.

Given an arbitrary homeomorphism $h \in \mathrm{Homeo}_+(S^1)$ we choose some neighbourhood U of $h^{-1}(0)$ such that the pre-image, under h, of all other fixed points of E, do not intersect $\overline{U}$. Then the restriction of Φ to the set $\mathcal{U} = \{h^{-1}Eh : h^{-1}(0) \in U\}$ is injective. Continuity of $\Psi|\mathcal{U}$ is merely a matter of inspection. To see why its inverse is continuous, fix again an arbitrary element h of $\mathrm{Homeo}_+(S^1)$ and some neighbourhood $\mathcal{V}$ of $\Psi(h)$. Fix some large k such that $\Phi(\tilde{h}) \in \mathcal{V}$ whenever $\tilde{h}^{-1}(p) = h^{-1}(p)$ for every periodic point p of period less than or equal to k. By labeling these points $p_1, \ldots, p_{|\ell^k - 1|}$, where $p_i = h^{-1}(i/\ell^k)$, and by choosing sufficiently small neighbourhoods $U_1, \ldots U_{|\ell^k-1|}$ of them, we guarantee that the (open) set

$$\mathcal{U}_0 = \{\tilde{h} \in \mathrm{Homeo}_+(S^1) : \tilde{h}^{-1}(i/\ell^k) \in U_i\}$$

is mapped by Φ into $\mathcal{V}$, proving that the inverse of $\Phi|\mathcal{U}$ is continuous. □

In virtue of Proposition 7.1, it is enough to prove the following: Fix an integer ℓ with $|\ell| \geq 2$ and let E denote the linear expanding circle map of degree ℓ. Then, given a generic orientation-preserving homeomorphism $h : S^1 \to S^1$, the map $f = h^{-1}Eh$ is wicked, meaning that the sequence $\sum_{k=0}^{n-1} h_*^{-1}E_*^k h_* m$ is dense in $\mathcal{M}_f(S^1)$. But h_* maps $\mathcal{M}_f$ homeomorphically onto $\mathcal{M}_E$, so it is indeed enough to prove that, for generic $h \in \mathrm{Homeo}_+(S^1)$, the sequence $\sum_{k=0}^{n-1} E_*^k h_* m$ is dense in $\mathcal{M}_E$. That is what we are going to do throughout the remainder of this section.

We identify the circle with the interval $I^0 = [0, 1)$. We denote by $\mathcal{A}$ the alphabet $\{0, \ldots, \ell - 1\}$ and write $\mathcal{A}^p$ for the set $\mathcal{A} \times \ldots \times \mathcal{A}$ (p times) of words of length p. If $\alpha \in \mathcal{A}^p$ and $\beta \in \mathcal{A}^q$ we denote by $\alpha\beta \in \mathcal{A}^{p+q}$ their

concatenation. That is, if $\alpha = 010$ and $\beta = 11$, then $\alpha\beta = 01011$. For each $k \in \mathbb{N}$ we partition I^0 into ℓ^k intervals $\{I^k_\alpha : \alpha \in \mathcal{A}^k\}$, where

$$I^p_\alpha = \left[\frac{\alpha}{\ell^k}, \frac{\alpha+1}{\ell^k}\right),$$

treating α as a natural number expressed in base ℓ. Thus if $\ell = 2$ and $k = 3$ we have $I^3_{000} = [0, \frac{1}{8})$, $I^3_{001} = [\frac{1}{8}, \frac{2}{8})$ and $I^3_{111} = [\frac{7}{8}, 1)$.

Every $h \in \mathrm{Homeo}_+(S^1)$ gives rise to a sequence of partitions $\mathcal{J}^k = \{J^k_\alpha : \alpha \in \mathcal{A}^k\}$, $k \in \mathbb{N}$, given by $J^k_\alpha = h^{-1}(I^k_\alpha)$. The sequence $\mathcal{J}^p$ (just like $\mathcal{I}^p$) is *consistent* in the following sense: for every $p, q \in \mathbb{N}$, and every $\alpha \in \mathcal{A}^p$, we have

$$J^p_\alpha = \bigcup_{\beta \in \mathcal{A}^q} J^{p+q}_{\alpha\beta}. \tag{40}$$

Note that $h_* m(I^p_\alpha) = m(J^p_\alpha)$. Moreover, for $q \geq 0$ we have

$$E^q_* h_*(I^p_\alpha) = m\left(\bigcup_{\beta \in \mathcal{A}^q} J^{q+p}_{\beta\alpha}\right) = \sum_{\beta \in \mathcal{A}^q} m(J^{q+p}_{\beta\alpha}). \tag{41}$$

Conversely, given any finite sequence of partitions $\mathcal{J}^1, \ldots, \mathcal{J}^q$ into, respectively, $\ell, \ldots, \ell^q$ intevals, consistent in the sense of (40), there exists a homeomorphism $h : S^1 \to S^1$ (e.g., a piecewise linear one) such that $h(J^j_\alpha) = I^j_\alpha$ for every $1 \leq j \leq q$ and $\alpha \in \mathcal{A}^j$.

To prove Theorem 3.9, we consider a countable base $\{V_i\}_{i\in\mathbb{N}}$ of the weak*-topology on $\mathcal{M}(S^1)$ where each V_i is of the form

$$V_i = \left\{\mu \in \mathcal{M}(S^1) : \left|\int \varphi^i_j \, d\mu - \int \varphi^i_j \, d\nu_i\right| < \epsilon^i_j \ \forall 1 \leq j \leq k_i\right\} \tag{42}$$

for some $\nu_i \in \mathcal{M}(S^1)$, $k_i \in \mathbb{N}$, continuous $\varphi^i_j : S^1 \to \mathbb{R}$ and $\epsilon^i_j > 0$.

Let $\mathcal{U}_{i,n}$ denote the (open) sets $\{h \in H(S^1) : \frac{1}{n}\sum_{k=0}^{n-1} E^k_* h_* m \in V_i\}$.

Lemma 7.2. *Suppose $V_i \cap \mathcal{M}_E(S^1) \neq$ and let m be any integer. Then $\bigcup_{n \geq m} \mathcal{U}_{i,n}$ is dense in $H(S^1)$.*

Once the above claim is proved, the proof of Theorem 3.9 follows by observing that $\frac{1}{n}\sum_{k=0}^{n-1} E^k_* h_* m$ accumulates on the whole of $\mathcal{M}_E(S^1)$ if and only if

$$h \in \bigcap_{\substack{i \in \mathbb{N} \text{ such that} \\ V_i \cap \mathcal{M}_E(S^1) \neq}} \bigcap_{m \geq 0} \bigcup_{n \geq m} \mathcal{U}_{i,n}. \tag{43}$$

Proof of Lemma 7.2. Fix $h \in H(S^1)$, V_i such that $V_i \cap \mathcal{M}_E(S^1) \neq$ and $\epsilon > 0$. The goal is to prove that if $n_0 \geq 1$ is sufficiently large, then for every $n > n_0$ there is some $h' \in H$ with $d(h', h) < \epsilon$, such that

$$E_*^k h'_* m \in V_i \quad \forall n_0 \leq k \leq n-1. \tag{44}$$

For then

$$\frac{1}{n}\sum_{k=0}^{n-1} E_*^k h'_* m = \frac{1}{n}\sum_{k=0}^{n_0-1} E_*^k h'_* m + \frac{1}{n}\sum_{k=n_0}^{n-1} E_*^k h'_* m \in V_i \tag{45}$$

provided that n is sufficiently large in comparison to n_0. To this end, let n_0 be any integer satisfying $\ell^{-n_0} < \epsilon$. Next pick some $\mu \in V_i \cap \mathcal{M}_E(S^1)$ and choose p large enough so that $\nu \in V_i$ whenever ν is a measure satisfying

$$\nu(I_\alpha^p) = \mu(I_\alpha^p) \quad \forall \alpha \in \mathcal{A}^p. \tag{46}$$

We define a consistent family of partitions $\{\mathcal{J}^k\}_{k=1}^{n-1}$ in the following manner: For $1 \leq k \leq n_0$ and $\alpha \in \mathcal{A}^k$ let $J_\alpha^k = h^{-1}(I_\alpha^k)$. For $n_0 < k \leq n-1$ and $\alpha \in \mathcal{A}^k$ write $\alpha = \beta\gamma$ with $\beta \in \mathcal{A}^{n_0}$ and $\gamma \in \mathcal{A}^{k-n_0}$. Define $\mathcal{J}^k$ in such a way that

$$m(J_{\beta\gamma}^k) = m(J_\beta^{n_0})\mu(I_\gamma^{k-n_0}). \tag{47}$$

Let $h' : S^1 \to S^1$ be a homeomorphism such that $h'(J_\alpha^k) = I_\alpha^k$ for every $\alpha \in \mathcal{A}^k$, $1 \leq k \leq n-1$. Then $d(h', h) < \epsilon$ since h and h' agree on each $J_\alpha^{n_0}$. Moreover (44) holds for such a choice of h'. Indeed, when $n_0 \leq k \leq n-1$ we may write $\beta \in \mathcal{A}^k$ as $\omega\tau \in \mathcal{A}^{n_0} \times \mathcal{A}^{k-n_0}$. Thus combining (41) and (47) we have

$$E_*^k h_* m(I_\alpha^p) = \sum_{\omega \in \mathcal{A}^{n_0}} \sum_{\tau \in \mathcal{A}^{k-n_0}} m(J_{\omega\tau\alpha}^{k+p}) \tag{48}$$

$$= \sum_{\omega \in \mathcal{A}^{n_0}} \sum_{\tau \in \mathcal{A}^{k-n_0}} m(J_\omega^{n_0})\mu(I_{\tau\alpha}^{k-n_0+p}) \tag{49}$$

$$= \sum_{\tau \in \mathcal{A}^{k-n_0}} \mu(I_{\tau\alpha}^{k-n_0+p}) = \mu(E^{-(k-n_0)}(I_\alpha^p)) = \mu(I_\alpha^p). \tag{50}$$

By our choice of p this implies that $E_*^k h'_* m \in V_i$ as required. □

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